On smoothness-asymmetric null infinities

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Abstract

We discuss the existence of asymptotically Euclidean initial data sets to the vacuum Einstein field equations which would give rise (modulo an existence result for the evolution equations near spatial infinity) to developments with a past and a future null infinity of different smoothness. For simplicity, the analysis is restricted to the class of conformally flat, axially symmetric initial data sets. It is shown how the free parameters in the second fundamental form of the data can be used to satisfy certain obstructions to the smoothness of null infinity. The resulting initial data sets could be interpreted as those of some sort of (non-linearly) distorted Schwarzschild black hole. Its developments would be so that they admit a peeling future null infinity, but at the same time have a polyhomogeneous (non-peeling) past null infinity.

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1 Introduction

This article is concerned with providing an example of asymptotically Euclidean, conformally flat initial data sets for the Einstein vacuum equations which are expected to have time developments for which the two disconnected parts of null infinity have different degrees of smoothness.

The analysis of the behaviour of the gravitational field in the region of spacetime near spatial infinity and null infinity carried out in [15, 26, 25] —for spacetimes with time reflexion symmetry— and in [27, 28] —for spacetimes without time reflexion— was motivated by the desire of setting on a sound footing *Penrose's proposal* for the description of the asymptotics of the gravitational field —see e.g. [20, 16, 18]. Penrose's proposal suggests that the asymptotic gravitational field of isolated systems should admit a *smooth conformal completion* at null infinity.

A large class of spacetimes satisfying Penrose's proposal has been constructed by Chruściel & Delay [6]. The examples provided rely in a refinement of a gluing construction introduced by Corvino [8] by means of which it is possible to construct time symmetric initial data sets which are essentially arbitrary inside a compact set, but are exactly Schwarzschild in the asymptotic region. More recently, a further development in the gluing techniques has allowed to extend Corvino's results to the non-time symmetric case, so that initial data sets which are arbitrary in a compact region can be glued to a stationary asymptotic region [5, 9]. As a result, all of the available examples of spacetimes satisfying Penrose's proposal are stationary in a spacetime neighbourhood of spatial infinity. In view of this, it is natural to ask whether initial data sets with stationary asymptotic regions are the only ones giving rise to spacetimes with a smooth null infinity. In relation to this latter issue, evidence for the following conjecture has been provided [26, 27]:

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Conjecture. *The time development of an asymptotically Euclidean initial data set which is conformally flat in a neighbourhood of infinity admits a conformal extension to both future and past null infinity of class C^k, with k a non-negative integer, if and only if the initial data are Schwarzschild to order p_* where $p_* = p_*(k)$ is a non-negative integer. If the development admits an extension of class C^∞, then the initial data have to be exactly Schwarzschild on $\mathcal{B}_a(i)$.*

Here, for *Schwarzschild up to order* p_* it is understood that asymptotic expansions of the initial data near infinity coincide up to order $\mathcal{O}(1/|y|^{p_*})$ with those of initial data for the Schwarzschild solution —see section 2 for more details. A similar conjecture is expected to hold for non-conformally flat initial data —so that the data should be stationary up to order p_* if a null infinity of class C^k is to be attained.

The calculations leading to the conjecture have shown the possibility of having a spacetime where future null infinity ($\mathscr{I}^+$) and past null infinity ($\mathscr{I}^-$) have different smoothness —note that the conjecture requires conditions on both parts of null infinity, in stark contrast with a previous version which arose in the analysis of time symmetric situations [26]. The possibility of having a spacetime where the two components of null infinity have different degrees of differentiability arose for the first time in the post-Newtonian analysis of the relativistic Kepler problem carried out by Walker & Will [29, 30]. Their calculations exhibit a system whose fall-off at null infinity is compatible with the *Peeling Behaviour* at $\mathscr{I}^+$, but not at $\mathscr{I}^-$. The latter phenomenon was regarded as a hint that requiring smoothness of the gravitational field at past null infinity may be a too stringent condition —see e.g. [22]. For many physical applications the presence of a non-smooth $\mathscr{I}^-$ may not be too hampering as long as one can ensure a peeling $\mathscr{I}^+$. And even if $\mathscr{I}^+$ happens to be non-smooth, most of the relevant structure at null infinity can still be recovered —see [7]. As pointed out by Penrose: "the issue is not whether regularity at $\mathscr{I}^+$ covers all situations we would like to call *asymptotically flat*; it is whether this regularity condition allows all the freedom that we need in order to describe isolated systems in general relativity" —see [21].

In [27], scripts in the computer algebra system `Maple V` were used to calculate asymptotic expansions of the time development of conformally flat (but non-time symmetric) initial data sets in a neighbourhood of spatial infinity. In particular, the components ϕ_j with $j = 0, \ldots, 4$, of the Weyl spinor ϕ_{ABCD} have an expansion of the form:

$$\phi_j \sim \sum_{p=|2-j|}^{\infty} \phi_j^{(p)} \rho^p,$$

where ρ is a geodesic distance, and the coefficients $\phi_k^{(p)}$ depend on τ, an affine parameter of conformal geodesics, and on some angular coordinates. In these coordinates $\tau = \pm 1$ corresponds to the locus of $\mathscr{I}^+$ and $\mathscr{I}^-$, respectively. It turns out that for $p = 0, \ldots, 4$, the coefficients $\phi_j^{(p)}$ extend smoothly through null infinity. However, from $p = 5$ onwards, one finds a series of *obstructions* to the smoothness of both future and past null infinity. For $p = 5$ the obstructions are time symmetric —that is, the obstructions for $\mathscr{I}^+$ and $\mathscr{I}^-$ are the same, but at $p = 6$ some of the obstructions do not possess this time symmetry. The latter —as it was discussed in previous paragraphs— opened the possibility of having a future and a past null infinity with different degrees of smoothness. A similar phenomenon occurs, as it is to be expected, if one analyses time asymmetric initial data sets which are not conformally flat. However, the overall picture is much more involved.

The calculations in [27, 28] assume the existence of initial data sets which are expandable in powers of $1/r$ near infinity. Existence proofs for this type of data have been provided in [12, 11]. However, the question whether the free parameters in the solution can be chosen such that the conditions leading to smoothness-asymmetric null infinities are satisfied remained open. This article provides an answer to this question. Ultimately, one would also like to make some assertion concerning the existence of the time development. This question is a much more difficult one and lies outside the scope of the present work.

The article is structured as follows. Section 2 is concerned with some aspects of conformally flat initial data sets which will be required in our investigations. In particular, some solutions to the momentum constraint more general than those of the Bowen-York Ansatz are considered, and

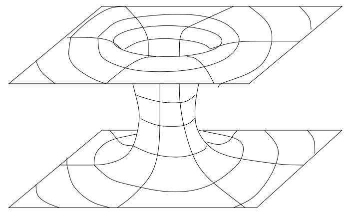

Figure 1: The topology of the initial hypersurface $\widetilde{\mathcal{S}}$ corresponds to that of time symmetric slices of Schwarzschild: two asymptotic regions connected by a "throat".

a general existence result for the class of initial data under consideration is recalled. In section 3 we discuss a certain class of obstructions to the smoothness of null infinity which allow us to construct initial data sets whose developments would have prescribed regularity at null infinity. Section 4 contains our main result and its proof.

The present work is a natural extension of the work carried out in [27, 28]. We have endeavour to follow to the nomenclature and notation of these references, however, for convenience we have made use of spin-weighted harmonics instead of certain unitary representations of $SU(2,\mathbb{C})$ when performing expansions. Accordingly, our notation has been adapted. The reader in need of a remainder on concepts and results on functional analysis (Sobolev spaces, Gâteaux derivatives, the implicit function theorem) is remitted to [1, 2, 19].

2 Framework

We shall restrict our discussion to solutions $(\widetilde{h}_{ab}, \widetilde{\chi}_{ab})$ to the vacuum Einstein constraint equations:

$$\widetilde{D}^a\widetilde{\chi}_{ab} - \widetilde{D}_b\widetilde{K} = 0, \tag{1}$$

$$\widetilde{r} - \widetilde{K}^2 + \widetilde{K}_{ab}\widetilde{K}^{ab} = 0, \tag{2}$$

on asymptotically Euclidean 3-manifolds, $\widetilde{\mathcal{S}}$, which are *maximal* —that is, $\widetilde{K} = 0$— and *conformally flat*. The metric $\widetilde{h}_{ab}$ will be taken to be *negative definite*. For simplicity, we shall also assume that the initial data sets are axially symmetric. Nevertheless, it must be pointed out that our analysis can be generalised to the axially symmetric and/or conformally flat settings. Furthermore, we shall assume that the 3-manifold $\widetilde{\mathcal{S}}$ has the topology of the time symmetric slice of the Schwarzschild solution: two asymptotically flat regions connected by a "throat —see figure 1".

The construction we want to implement depends crucially on the properties of the solutions to the constraints in a neighbourhood of infinity. Therefore, we adopt the conformal compactification picture. Let $\mathcal{S}$ be a smooth, orientable, connected, compact 3-manifold —for definiteness, we can think of S^3. In $\mathcal{S}$ we shall consider two special points i and i' —say, the North and South poles of S^3— which will represent the two infinities of the asymptotic ends of the physical data $(\widetilde{\mathcal{S}}, \widetilde{h}_{ab}, \widetilde{\chi}_{ab})$, and so that $\widetilde{S} = S \setminus \{i, i'\}$.

2.1 The conformal method

Following the standard approach —*the Conformal Method*— we set

$$\widetilde{h}_{ab} = \vartheta^4 h_{ab}, \quad \widetilde{\chi}_{ab} = \vartheta^{-2}\psi_{ab}. \tag{3}$$

Let x^a denote h-normal coordinates centred at i. When no confusion arises, we may write x instead of x^a. In order to ensure the asymptotic Euclideanity of the initial data we require

$$\psi_{ab} = \mathcal{O}(|x|^{-4}) \text{ as } |x| \to 0, \tag{4a}$$

$$\lim_{|x|\to 0} |x|\vartheta = 1, \tag{4b}$$

with $|x|^2 = \delta_{ab}x^a x^b$. Similarly, if y^a denote h-normal coordinates centred at i', we shall require that

$$\psi_{ab} = \mathcal{O}(|y|^{-4}) \text{ as } |y| \to 0, \tag{5a}$$
$$\lim_{|y|\to 0} |y|\vartheta = c', \tag{5b}$$

where c' is a positive constant.

Because of the conformal flatness of the data, we set in a neighbourhood of i, $h_{ab} = -\delta_{ab}$, so that near i the constraint equations reduce to:

$$\partial_a \psi^{ab} = 0, \tag{6a}$$
$$\Delta\vartheta = -\frac{1}{8}\psi_{ab}\psi^{ab}\vartheta^{-7}, \tag{6b}$$

where Δ denotes the flat Laplacian.

2.2 Solutions to the momentum constraint

The solutions to the momentum constraint in Euclidean space have been extensively studied in the literature —see e.g. [12, 3, 27]. For convenience, we will use the version given in [27], in which the solutions to the momentum constraint are derived using the space-spinor formalism —see [24, 15]. For this, we consider the vector field $e^a_{(3)}$ which in Cartesian coordinates is given by $x^a/|x|$, and complete it to an orthonormal basis $\{e^a_{(1)}, e^a_{(2)}, e^a_{(3)}\}$. Let $\psi_{(i)(j)} = \psi_{ab}e^a_{(i)}e^b_{(j)}$ denote the components of the second fundamental form ψ_{ab} with respect to this basis. In a space-spinor formalism, and under the assumption of a maximal slice, the tensor ψ_{ab} will be represented by means of a totally symmetric spinor ψ_{ABCD} which is related to $\psi_{(i)(j)}$ via the spatial Infeld-van der Waerden symbols $\sigma^{AB}_{(i)}$:

$$\psi_{(i)(j)} = \sigma^{AB}_{(i)}\sigma^{CD}_{(j)}\psi_{ABCD}. \tag{7}$$

The spinor ψ_{ABCD} being totally symmetric can be decomposed as

$$\psi_{ABCD} = \psi_0\epsilon^0_{ABCD} + \psi_1\epsilon^1_{ABCD} + \psi_2\epsilon^2_{ABCD} + \psi_3\epsilon^3_{ABCD} + \psi_4\epsilon^4_{ABCD}, \tag{8}$$

where the spinors ϵ^i_{ABCD}, $i = 0, \ldots, 4$ are also totally symmetric and

$$\epsilon^k_{(ABCD)_j} = \delta^k_j \binom{4}{k}^{-1}, \tag{9}$$

the subindex $j = 0, \ldots, 4$ in $_{(ABCD)_j}$ indicating that j of the indices have to be set equal to 1. The particular solution to the momentum constraint that we are going to consider can be split in six independent pieces:

$$\psi_{ABCD} = \psi^{\mathcal{A}}_{ABCD} + \psi^{\mathcal{J}}_{ABCD} + \psi^{-2,2}_{ABCD} + \psi^{-1,2}_{ABCD} + \psi^{-1,3}_{ABCD} + \psi^{0,2}_{ABCD}. \tag{10}$$

The term $\psi^{\mathcal{A}}_{ABCD}$ associated with the expansion conformal Killing vector —it gives rise, for example, to time asymmetric slices in Schwarzschild [4, 13, 23]. The term $\psi^{\mathcal{J}}_{ABCD}$ encodes the angular momentum content of the data. While the remaining terms are diverse multipolar terms —the first superindex denotes the decay of the term ($|x|^{-2}$, $|x|^{-1}$ or $|x|^0$) and the second its multipolar nature: quadrupolar (2^2) or octupolar (2^3). For further discussion see [26].

In the case of the $\psi^{\mathcal{A}}_{ABCD}$ term, one has a single non-vanishing component:

$$\psi^{\mathcal{A}}_2 = -\frac{\mathcal{A}}{|x|^3}. \tag{11}$$

Assuming axial symmetry, the non-vanishing components of $\psi^{\mathcal{J}}_{ABCD}$ are given by:

$$\psi^{\mathcal{J}}_1 = \frac{12}{|x|}\sqrt{\frac{\pi}{5}}\mathrm{i}\mathcal{J}\;{}_1Y_{20}, \quad \psi^{\mathcal{J}}_3 = -\frac{12}{|x|}\sqrt{\frac{\pi}{5}}\mathrm{i}\mathcal{J}\;{}_{-1}Y_{20}. \tag{12}$$

For the remaining terms one has that:

$$\psi_j^{n,q} = (-\mathrm{i})^{j+6-2q}\sqrt{\frac{4\pi}{2q+1}}L_{j,n;q}\ {}_{2-j}Y_{q0}\rho^n, \tag{13}$$

for $j = 0, \ldots, 4$. The coefficients $L_{0,n;q}$ and $L_{4,n;q}$ are freely specifiable complex numbers satisfying the reality condition

$$L_{0,n;q} = \overline{L}_{4,n;q}. \tag{14}$$

The remaining coefficients are calculated via the formulae:

$$L_{1,n;q} = \frac{\big((L_{4,n;q} - L_{0,n;2q})(q+1)q + 4(n+7)^2 L_{0,n;q}\big)}{(n+7)(2(n+8)(n+6) - (q+2)(q-1))}\sqrt{(q+2)(q-1)}, \tag{15a}$$

$$L_{2,n;q} = \frac{3(L_{0,n;q} + L_{4,n;q})}{2(n+7)^2 - q(q+1)}\sqrt{(q+2)(q+1)q(q-1)}, \tag{15b}$$

$$L_{3,n;q} = \frac{\big((L_{0,n;q} - L_{4,n;2q})(q+1)q + 4(n+7)^2 L_{4,n;q}\big)}{(n+7)(2(n+8)(n+6) - (q+2)(q-1))}\sqrt{(q+2)(q-1)}. \tag{15c}$$

In addition, the following *regularity conditions* will be required to hold:

$$L_{0,-2;2} + L_{4,-2;2} = 0, \quad L_{0,-1;3} + L_{4,-1;3} = 0. \tag{16}$$

The latter, together with (14) that $L_{0,-2;2}$, $L_{4,-2;2}$, $L_{0,-1;3}$ and $L_{4,-1;3}$ are pure imaginary. The conditions (16) are intended to preclude the appearance of a certain type of logarithmic divergences at the sets where null infinity touches spatial infinity —see [15, 27], and also the discussion in section 3.

From the point of view of the normal coordinates y^a centred on i', the different terms of ψ_{ABCD} have an analogous decay to the one near i:

$$\psi^{\mathcal{A}}_{ABCD} = \mathcal{O}(|y|^{-3}), \quad \psi^{\mathcal{J}}_{ABCD} = \mathcal{O}(|y|^{-3}), \quad \psi^{n,k}_{ABCD} = \mathcal{O}(|y|^{n}). \tag{17}$$

Thus, all the terms —save for $\psi^{0,2}_{ABCD}$— are singular at both i and i' —see [12].

For latter use, it is noted that if ψ_{ab} is of the form given by equation (10) then:

$$\psi_{ab}\psi^{ab} = \psi_{ABCD}\psi^{ABCD} = \frac{1}{|x|^6}\left(\frac{\mathcal{A}^2}{6} - 12\mathcal{J}^2 + 24\sqrt{\frac{2\pi}{5}}\mathcal{J}^2 Y_{20}\right) + \mathcal{O}\left(\frac{1}{|x|^5}\right). \tag{18}$$

Note that the above expansion truncates at order $\mathcal{O}(1/|x|^2)$, the first contributions of the higher multipolar terms in $\psi^{n,k}_{ab}$ appearing at order $\mathcal{O}(1/|x|^5)$. Also, $\psi_{ab}\psi^{ab} < 0$. Finally, we note that for a neighbourhood $\mathcal{B}_a(i)$ of radius a centred at i, one has that

$$|x|^8\psi_{ab}\psi^{ab} \in E^\infty(\mathcal{S}\setminus\{i'\}), \tag{19}$$

where $E^\infty(\mathcal{B}) = \{f + |x|g \mid f, g \in C^\infty(\mathcal{B})\}$. Similar behaviour is obtained on $\mathcal{S}\setminus\{i\}$ for $|y|^8\psi_{ab}\psi^{ab}$.

2.3 General existence results

The assumption on the conformal flatness of the 3-metric $\widetilde{h}_{ab}$ and the choice of solutions to the momentum constraint made in the previous section fulfil the hypothesis of the existence results obtained in [12]. Thus, one has:

Theorem 1 (Dain & Friedrich, 2001). *If $\widetilde{h}_{ab}$ is conformally flat, and $\widetilde{\chi}_{ab}$ is given via equation (10), then there is a unique, positive solution ϑ to the Licnerowicz equation (6b) with the boundary conditions (4b) and (5b). Moreover, by virtue of (19) one has that on $\mathcal{S}\setminus\{i'\}$:*

$$\vartheta = \frac{1}{|x|} + W, \tag{20}$$

where $W \in E^\infty(\mathcal{S}\setminus\{i'\})$.

Because of $W \in E^\infty(\mathcal{S} \setminus \{i'\})$, the function W can be expanded in a neighbourhood of i as:

$$W = \frac{m}{2} + \big(d_a x^a + m'|x|\big) + \big(q_{ab}x^a x^b + d'_a x^a |x|\big) + \big(o_{abc}x^a x^b x^c + q'_{ab}x^a x^b |x|\big) + \mathcal{O}(|x|^4), \quad (21)$$

where m is the ADM mass of the initial data, and m', d_a and d'_a, q_{ab} and q'_{ab}, o_{abc} are, respectively, monopolar, dipolar, quadrupolar and octupolar terms. In reference [26] it has been shown that, without loss of generality, one can make use of the translational freedom in the setting to remove the dipolar term, d_a, in the previous expansion. Proceeding in such a way, we are left with

$$W = \frac{m}{2} + m'|x| + \big(q_{ab}x^a x^b + d'_a x^a |x|\big) + \big(o_{abc}x^a x^b x^c + q'_{ab}x^a x^b |x|\big) + \mathcal{O}(|x|^4). \qquad (22)$$

The explicit calculations carried out in [27] show that in fact $m' = 0$ and $d'_a = 0$.

Alternatively, one could have used the spherical coordinates associated with the normal coordinates x^a. Due to our assumption on axial symmetry the expansions are given by[1] :

$$W = \frac{m}{2} + \frac{1}{2!}r^2 \sum_{l=0}^{2} \omega_{2;l} Y_{l0} + \frac{1}{3!}r^3 \sum_{l=0}^{3} \omega_{3;l} Y_{l0} + \mathcal{O}(r^4), \qquad (23)$$

where $r = |x|$, and $\omega_{p;l,m} \in \mathbb{R}$.

The function W satisfies the elliptic equation:

$$\Delta W = -\frac{1}{8}\psi_{ab}\psi^{ab}(1/|x| + W)^{-7}. \qquad (24)$$

Thus, the coefficients $\omega_{2;l,m}$ and $\omega_{3;l,m}$ are determined by the boundary conditions (4a)-(5b) and the parameters of ψ_{ab} —that is, $\mathcal{A}$, $\mathcal{J}$, $L_{0,n;k}$ and $L_{4,n;k}$. Thus, sometimes we shall write $W = W[\mathcal{A}, J, L](x)$. The elliptic nature of (24) reveals that W contains information of global nature. The local part —around infinity— is encoded in the $1/|x|$ term of (20) corresponding to the conformal flatness of the data.

From equation (24) and taking into account the boundary conditions (4a)-(4b) one gets the integral representation:

$$W = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\psi_{ab}(x')\psi^{ab}(x')}{8(1/|x'| + W(x'))^7}\frac{1}{|x - x'|}dx'. \qquad (25)$$

The function $-1/4\pi|x - x'|$ corresponds to the Green's function of the Laplacian on $\mathbb{R}^3$.

For later use, we show that the function $W = W[\mathcal{A}, J, L](x)$ depends —at least— in a C^1 fashion on the parameters $\mathcal{A}$, $\mathcal{J}$, $L_{0,n;k}$ and $L_{4,n;k}$ which determine the tensor ψ_{ab}.

Lemma 1. *The solution W of equation (24), with ψ_{ab} given by equation (10) is C^1 in the parameters $\mathcal{A}$, $\mathcal{J} \in \mathbb{R}$ and $L_{0,n;k}$, $L_{4,n;k} \in \mathbb{C}$*

To see this, we rewrite equation (24) as

$$\mathcal{G}(\mathcal{J}; W) = \Delta W + \frac{1}{8}\psi_{ab}\psi^{ab}(1/|x| + W)^{-7} = 0, \qquad (26)$$

and concentrate on the dependence of W with respect to $\mathcal{J}$ —the analysis with respect to the other parameters of ψ_{ab} is similar. The mapping $\mathcal{G} : \mathbb{R}\times W^{2,p}(\mathcal{S}) \longrightarrow L^p(\mathcal{S})$, $p \geq 2$, where $W^{2,p}(\mathcal{S})$ denotes the Sobolev space of functions on $\mathcal{S}$ with L^p-integrable weak second order derivatives, is

[1] In reference [27], on which the present work is based, a certain type of unitary representations of $SU(2,\mathbb{C})$, $T_m{}^k{}_j$, is used. Here,instead, we make use of the more standard spin-weighted spherical harmonics ${}_sY_{nm}$. The relation between the two sets of functions is given by

$${}_sY_{nm} \mapsto (-\mathrm{i})^{s+2n-m}\sqrt{\frac{2n+1}{4\pi}}T_{2n}{}^{n-m}{}_{n-s}.$$

Accordingly, we have slightly changed our notation with respect to that of [27].

at least a C^2 mapping (in the L^p norm) between Banach spaces —actually, it can be shown to be C^∞, but C^2 will be enough for our applications. That $\mathcal{G} \in C^0(\mathbb{R}\times W^{2,p}, L^p)$ can be shown using

$$\begin{aligned}\|\mathcal{G}(\mathcal{J},W)-\mathcal{G}(\mathcal{J}_0,W_0)\|_{L^p} \leq \|\Delta W-\Delta W_0\|_{L^p} \\ +C\|(1+|x|W)^{-7}\|_{L^p}\||x|^7\psi_{ab}\psi^{ab}(\mathcal{J})-|x|^7\psi_{ab}\psi^{ab}(\mathcal{J}_0)\|_{L^p} \\ +C\||x|^7\psi_{ab}\psi^{ab}(\mathcal{J}_0)\|_{L^p}\|(1+|x|W)^{-7}-(1+|x|W_0)^{-7}\|_{L^p},\end{aligned} \quad (27)$$

and noting that ΔW converges to ΔW_0 is W goes to W_0 as $W^{2,p}$ is a Banach space, and that $\psi_{ab}\psi^{ab}$ is in our case analytic function of $\mathcal{J}$. To establish the C^1 character of the map $\mathcal{G}$ one has to perform a similar discussion with the Gâteaux derivative of $\mathcal{G}$, $\mathrm{d}_G\mathcal{G} : \mathbb{R}^2 \times W^{2,p}\times W^{2,p} \longrightarrow L^{2,p}$, which is given by

$$\mathrm{d}_G\mathcal{G}(\mathcal{J},j;W,w) = \Delta w - \frac{1}{8}j\partial_{\mathcal{J}}\psi_{ab}\psi^{ab}(1/|x|+W)^{-7} + \frac{7}{8}w\psi_{ab}\psi^{ab}(1/|x|+W)^{-8}. \quad (28)$$

Again, the analyticity of ψ_{ab} with respect to $\mathcal{J}$ is crucial to establish the continuity of $\mathrm{d}_G\mathcal{G}$. Higher order derivatives are dealt with in a similar fashion.

Now, let us consider a solution W_0 to equation (24), corresponding to a certain value, $\mathcal{J}_0$, of the angular momentum, so that

$$\mathcal{G}(\mathcal{J}_0,W_0)=0. \quad (29)$$

The existence of such a solution is guaranteed by the theorem 1. The linearised operator

$$\mathcal{L}w = \Delta w - \frac{7}{8}\psi_{ab}\psi^{ab}(1/|x|+W)^{-8}w, \quad (30)$$

$\mathcal{L}: W^{2,p}\longrightarrow L^p$ is an isomorphism: due to $\psi_{ab}\psi^{ab}>0$ the only solution to $\mathcal{L}w=0$ is $w=0$ (seen using integration by parts); further, $\mathcal{L}$ is selfadjoint and, thus, using the Fredholm alternative it is also surjective —see eg. [12]. So, from the implicit function theorem we know there exists solution $W=W[\mathcal{J}](x)$ to equation (26) for $\mathcal{J}$ sufficiently close to $\mathcal{J}_0$ —cfr. [1]. By unicity, this solution has to coincide with that one given in theorem 1. More importantly for our purposes is that $W\in C^2(U,X)$ —actually C^∞!— for some $U\subset\mathbb{R}$, $\mathcal{J}_0\in U$ and $X\subset W^{2,p}$, $W(\mathcal{J}_0)\in X$. Poinwise differentiability with respect to $\mathcal{J}$ follows from the Sobolev embedding theorem: if $(k-r-\alpha)/n\geq 1/p$ then $W^{k,p}(U)\subset C^{r,\alpha}(U)$, with $U\subset\mathbb{R}^n$ —see, for example [2]. In our case $k=2$ (at least!), $n=1$, $r\geq 1$ and p suitably large —say, $p\geq 2$. Note that if one only had $W\in C^1(U,X)$ then the pointwise continuity —with respect to $\mathcal{J}$— of $\partial_{\mathcal{J}}W$ can not be concluded.

3 Obstructions to the smoothness of null infinity

As mentioned in the introduction, the calculations carried out in [27] provide asymptotic expansions for the components of the Weyl spinor which allow to relate the structure of the initial data with the radiative properties of the development. Here, and in what follows we will concentrate our attention on the null infinity associated with the asymptotic end i. What happens at the null infinity assigned to the second asymptotic end, i', will not be of relevance for our purposes. Assuming axial symmetry, the components of the Weyl tensor can be written as:

$$\phi_j = \sum_{p=0}\sum_{l=|2-j|}^{p}\alpha_{j,p;l}(\tau)r^p \ {}_{2-j}Y_{l0}, \quad (31)$$

where $j=0,\dots,4$. The coefficients $\alpha_{j,p;l}(\tau)$ for $p=0,\dots,4$ are *polynomial* in τ.

For $p=5$, they are polynomial in τ for $l=0,1,3$. However, for $l=2$ one has that

$$\alpha_{j,5;2}(\tau) = \Upsilon_{5;2}\left((1-\tau)^{7-j}\mathcal{P}_j(\tau)\ln(1-\tau)+(1+\tau)^{3+j}\mathcal{P}_{4-j}(\tau)\ln(1+\tau)\right)+\mathcal{Q}(\tau), \quad (32)$$

where $\mathcal{P}_j(\tau)$, $\mathcal{P}_{4-j}(\tau)$ and $\mathcal{Q}(\tau)$ are some polynomials in τ. In particular, $\mathcal{P}_j(\pm1)\neq 0$, $\mathcal{P}_{4-j}(\pm 1)\neq 0$.

Similarly, for $p=6$ the coefficients $\alpha_{j,p;l}(\tau)$ are polynomial for $p=0,1,4$, while the coefficients of the $l=2$ harmonics are of the form

$$\alpha_{j,6;2} = \Upsilon^{+}_{6;2}(1-\tau)^{8-j}\ln(1-\tau)\mathcal{P}_j(\tau) + \Upsilon^{-}_{6;2}(1+\tau)^{4+j}\mathcal{P}_{4-j}(\tau)\ln(1+\tau) + \mathcal{Q}(\tau), \tag{33}$$

where again $\mathcal{P}_j(\tau)$, $\mathcal{P}_{4-j}(\tau)$ and $\mathcal{Q}(\tau)$ are some polynomials in τ, $\mathcal{P}_j(\pm 1)\neq 0$, $\mathcal{P}_{4-j}(\pm 1)\neq 0$ —different to those in equation (32). Finally,

$$\alpha_{j,6;3}(\tau) = \Upsilon_{6;3}\left((1-\tau)^{8-j}\mathcal{P}_{j+1}(\tau)\ln(1-\tau) + (1+\tau)^{4+j}\mathcal{P}_{5-j}(\tau)\ln(1+\tau)\right) + \mathcal{Q}(\tau). \tag{34}$$

The coefficients $\alpha_{j,p;q}(\tau)$ for $p\geq 7$ are expected to exhibit a similar pattern. In particular, it is conjectured that there is an infinite hierarchy of coefficients $\Upsilon^{\pm}_{p;q}$ associated with logarithmic terms of the form discussed above. These details will not be relevant for our analysis.

The coefficients $\Upsilon_{5;2}$, $\Upsilon^{\pm}_{6;2}$ and $\Upsilon_{6;3}$ —*the obstructions to the smoothness of null infinity*— are given in terms of the freely specifiable data by:

$$\Upsilon_{5;2} = 9\sqrt{\frac{5}{\pi}}m^2\omega_{2;2,2} + \frac{37602}{199}m\mathcal{J}^2 - \frac{3099}{199}\sqrt{6}m^2\mathrm{i}\alpha + \frac{2046}{199}\sqrt{6}m\mathrm{i}\beta_I + \frac{448}{199}\sqrt{6}\mathrm{i}\delta_I, \tag{35}$$

$$\begin{aligned}\Upsilon^{\pm}_{6;2} = {}& \frac{2198208}{6965}\mathcal{J}^2 \pm \frac{62451}{14}\mathcal{A}\mathcal{J}^2 - \frac{62691}{2408}\sqrt{6}\mathrm{i}\mathcal{A}\beta_I + \frac{1791}{14}\sqrt{6}\beta_R \pm \frac{263327}{31605}\mathrm{i}\beta_I \\ & -\frac{116}{43}\sqrt{6}\mathrm{i}\mathcal{A}\delta_I - \frac{1791}{7}\sqrt{6}\delta_R \mp \frac{183184}{13545}\sqrt{6}\mathrm{i}\delta_I,\end{aligned} \tag{36}$$

$$\Upsilon_{6;3} = 12\sqrt{\frac{7}{\pi}}\omega_{3;3,3} - \frac{565753248}{82585}\mathrm{i}\mathcal{J}^3 + \frac{36399}{415}\sqrt{6}\mathcal{J}\beta_I + \frac{3808}{415}\sqrt{6}\mathcal{J}\delta_I - \frac{7272}{415}\sqrt{30}\mathrm{i}\gamma. \tag{37}$$

where in order to render the above expressions more readable we have set

$$L_{0,-2;4} = \mathrm{i}\alpha, \qquad L_{4,-2;4} = -\mathrm{i}\alpha, \tag{38a}$$
$$L_{0,-1;4} = \beta_R + \mathrm{i}\beta_I, \qquad L_{4,-1;4} = \beta_R - \mathrm{i}\beta_I, \tag{38b}$$
$$L_{0,-1,6} = \mathrm{i}\gamma, \qquad L_{4,-1;6} = -\mathrm{i}\gamma, \tag{38c}$$
$$L_{0,0;4} = \delta_R + \mathrm{i}\delta_I, \qquad L_{4,0;4} = \delta_R - \mathrm{i}\delta_I, \tag{38d}$$

in accordance with the reality conditions (14) and the regularity conditions (16).

The role of the obstructions (35)-(37) can be better understood by noting that the conformal factor Θ —rendering the representation of the region of spacetime near null and spatial infinity in which the components of the Weyl tensor (31) are calculated— is of the form,

$$\Theta = \frac{D_a\vartheta D^a\vartheta}{\vartheta^3}(1-\tau^2), \tag{39}$$

where $D_a\vartheta D^a\vartheta/\vartheta^3 = \mathcal{O}(\rho)$. The locus of null infinity is given by $\mathscr{I}^{\pm} = \{\tau=\pm 1, r\neq 0\}$. In addition, we define $I = \{\rho = 0, |\tau|<1\}$ and $I^{\pm} = \{\rho=0, \tau=\pm 1\}$ corresponding, respectively, to the *cylinder at spatial infinity* and the *critical sets* where null infinity "touches" spatial infinity. Thus, the coefficients $\alpha_{j,p;q}(\tau)$ in the expansion (31), considered as functions on I diverge on the critical sets $I^{\pm}$ if the relevant obstructions are not satisfied. Because of the hyperbolic nature of the equations governing the evolution of the gravitational field, one expects that these singularities will propagate along null infinity.

3.1 A particular choice of the data

Inspired by the above discussion, we are interested in constructing an initial data such that:

$$\Upsilon_{5;2} = \Upsilon^{+}_{6;2} = \Upsilon_{6;3} = 0, \tag{40a}$$
$$\Upsilon^{-}_{6;2} \neq 0. \tag{40b}$$

The evolution of such an initial data set will possess —modulo an existence proof for the conformal field equations which is valid up to the critical sets, $I^{\pm}$, which is not yet available, see e.g. [17]—

a null infinity where $\mathscr{I}^+$ and $\mathscr{I}^-$ have different smoothness. Assuming that the conditions (40a) and (40b) are sharp, a rough count using (32)-(34) suggests that the time development of our initial data will have an $\mathscr{I}^-$ which, at best, will be of class C^3 and an $\mathscr{I}^+$ of at least class C^4. In particular it is claimed the following:

Conjecture. *The time development of an initial data set $(\widetilde{h}_{ab}, \widetilde{\chi}_{ab})$ such that the conditions (40a) and (40b) hold admits a peeling future null infinity and a polyhomogeneous past null infinity. More precisely, using and an adapted gauge the components of the Weyl tensor*[2] *decay, near $\mathscr{I}^+$, as*

$$\phi_0 = \mathcal{O}(1/r), \quad \phi_1 = \mathcal{O}(1/r^2), \quad \phi_2 = \mathcal{O}(1/r^3), \quad \phi_3 = \mathcal{O}(1/r^4), \quad \phi_4 = \mathcal{O}(1/r^5). \tag{41}$$

While near $\mathscr{I}^-$ the decay will be

$$\phi_4 = \mathcal{O}(1/r), \quad \phi_3 = \mathcal{O}(1/r^2), \quad \phi_2 = \mathcal{O}(1/r^3), \quad \phi_1 = \mathcal{O}(1/r^4), \quad \phi_0 = \mathcal{O}(\ln r/r^5). \tag{42}$$

That is, the resulting spacetime is expected to peel at $\mathscr{I}^+$, but not at $\mathscr{I}^-$. For a discussion of the notion of polyhomogeneity at null infinity and non-peeling spacetimes see for example [7].

4 The main result

In this section it is shown how the freely specifiable parameters contained in the initial data discussed in section 2 —the scalars $\mathcal{A}$, $\mathcal{J}$ and α, β_R, β_I, γ, δ_R and δ_I—can be used to construct initial data satisfying the conditions (40a) and (40b). To this end, one has to find a way of controlling the coefficients $\omega_{2;2}$ and $\omega_{3;3}$ in the expansion (23).

Theorem 2. *Given $\mathcal{A}_0 \in \mathbb{R}$, $\mathcal{A}_0 \neq 0$ and a conformally flat initial data set with second fundamental form given by (10) and satisfying the boundary conditions (4b) and (5b), there exists a neighbourhood $\mho \subset \mathbb{R}^3$, $(0,0,0) \in \mho$ such that if $(\mathcal{J}, \beta_R, \beta_I) \in \mho$, then there are unique $\mathcal{A} = \mathcal{A}(\mathcal{J}, \beta_R, \beta_I)$, $\alpha = \alpha(\mathcal{J}, \beta_R, \beta_I)$, $\delta_R = \delta_R(\mathcal{J}, \beta_R, \beta_I)$, $\delta_I = \delta_I(\mathcal{J}, \beta_R, \beta_I)$, $\gamma = \gamma(\mathcal{J}, \beta_R, \beta_I)$ with $\mathcal{A}(0,0,0) = \mathcal{A}_0$, $\alpha(0,0,0) = \delta_R(0,0,0) = \delta_I(0,0,0) = \gamma(0,0,0) = 0$, such that*

$$\Upsilon_{5;2} = \Upsilon^+_{6;2} = \Upsilon_{6;3} = 0,$$

The condition

$$\Upsilon^-_{6;2} \neq 0,$$

is satisfied if $\mathcal{J} \neq 0$, $\beta_R \neq 0$, $\beta_I \neq 0$.

Thus, given a particular time asymmetric, conformally flat Schwarzschild initial data set, there is always a (non-linear) perturbation of it —actually an open set of them— satisfying the conditions (40a) and (40b). Note that these perturbations have necessarily non-vanishing angular momentum; furthermore, from a careful analysis of the proof it stems that all the parameters of the second fundamental form have to be non-zero —justifying the somewhat complicated-looking choice of the extrinsic curvature made in 2.2. The different parts of the proof will be discussed in the sequel.

4.1 Integral representations of $\omega_{2;2}$ and $\omega_{3;3}$.

Our first task is to obtain some expressions which enable us to control the expansion coefficients in (23). The explicit calculations of [27] show that:

$$W = \frac{m}{2} + \frac{1}{2!} r^2 \omega_{2;2} Y_{20} + \frac{1}{3!} r^3 \sum_{l=0}^{3} \omega_{3;l} Y_{lm} + \mathcal{O}(r^4), \tag{43}$$

[2]The convention used here to denote the different components of the Weyl tensor is exactly the opposite of the standard Newman-Penrose (NP) convention. That is, our ϕ_0 corresponds to NP's ψ_4, and so on.

in $\mathcal{S}\setminus\{i'\}$, where

$$\omega_{3;0} = -2\sqrt{\pi}\left(\frac{3J^2}{4}+\frac{\mathcal{A}^2}{96}\right), \tag{44a}$$

$$\omega_{3;1} = 3\sqrt{\frac{\pi}{5}}J^2, \tag{44b}$$

$$\omega_{3;2} = 0. \tag{44c}$$

On the other hand, the coefficients $\omega_{2;2}$ and $\omega_{3;3}$ do not admit such closed and neat expressions. They contain information of global nature. Thus, in order to control them we have to resort to the integral representation (25).

We begin by noting that:

$$\partial_{rr}W = \omega_{2;2}Y_{20} + \mathcal{O}(r). \tag{45}$$

In order to differentiate the integral representation (25) we have to proceed with care, for the Green function $1/|x-x'|$ is singular if $x=x'$. We begin by letting

$$F(x) = \frac{\psi_{ab}(x)\psi^{ab}(x)}{(1/|x|+W(x))^7}. \tag{46}$$

Due to the existence results of Theorem 1, $F(x)\in E^\infty(\mathcal{S}\setminus\{i'\})$, and $F(x)=\mathcal{O}(|x|)$. Accordingly, $\partial_r^{(k)}F$, $k=1,2,3$, are regular at $r=|x|=0$. And so,

$$\begin{aligned} W &= \frac{1}{32\pi}\int_{\mathbb{R}^3} F(x')\frac{1}{|x-x'|}d^3x', \\ &= \frac{1}{32\pi}\int_{\mathbb{R}^3} F(x-x')\frac{1}{|x'|}d^3x'. \end{aligned} \tag{47}$$

Frome where

$$\partial_r^{(k)}W = \frac{1}{32\pi}\int_{\mathbb{R}^3}\partial_r^{(k)}F(x-x')\frac{1}{|x'|}d^3x', \tag{48}$$

with $k=1,2,3$. Consequently, we find that

$$\omega_{2;2} = \frac{1}{32\pi}\int_{\mathbb{R}^3}\langle Y_{20}|\partial_{rr}F(-x')\rangle\frac{1}{|x'|}d^3x', \tag{49}$$

where

$$\langle Y_{lm}|H(\theta,\varphi)\rangle = \int_0^{2\pi}\int_0^{\pi}\overline{Y}_{lm}(\theta,\varphi)H(\theta,\varphi)\sin\theta d\theta d\varphi, \tag{50}$$

denotes the component of H on the (l,m)-harmonic. Similarly, one finds that

$$\omega_{3;3} = \frac{1}{32\pi}\int_{\mathbb{R}^3}\langle Y_{30}|\partial_{rrr}F(-x')\rangle\frac{1}{|x'|}d^3x'. \tag{51}$$

4.2 Satisfying the conditions for a "smoothness asymmetric null infinity".

In order to see how the conditions (40a) and (40b) can be satisfied, we begin by considering imaginary parts of the obstructions (35)-(37):

$$\mathrm{Im}(\Upsilon_{5;2}) = -\frac{3099}{199}\sqrt{6}m^2\alpha + \frac{2046}{199}\sqrt{6}m\beta_I + \frac{448}{199}\sqrt{6}\delta_I = 0, \tag{52a}$$

$$\mathrm{Im}(\Upsilon^+_{6;2}) = -\frac{62691}{2408}\sqrt{6}\mathcal{A}\beta_I + \frac{263327}{31605}\beta_I - \frac{116}{43}\sqrt{6}\mathcal{A}\delta_I - \frac{183184}{13545}\sqrt{6}\delta_I = 0, \tag{52b}$$

$$\mathrm{Im}(\Upsilon_{6;3}) = -\frac{565753248}{82585}\mathcal{J}^3 - \frac{7272}{415}\sqrt{30}\gamma = 0. \tag{52c}$$

The condition (52a) can be used to solve for α in terms of β_I and δ_I. Condition (52b) can be used to solve for δ_I in terms of $\mathcal{A}$ and β_I. Finally, condition (52c) can be used to write γ in terms of $\mathcal{J}$, rendering $\gamma \propto \mathcal{J}^3$. On similar lines

$$\mathrm{Re}(\Upsilon^+_{6;2}) = \frac{2198208}{6965}\mathcal{J}^2 + \frac{62451}{14}\mathcal{A}\mathcal{J}^2 + \frac{1791}{14}\sqrt{6}\beta_R - \frac{1791}{7}\sqrt{6}\delta_R = 0, \tag{53}$$

can be used to solve for δ_R in terms of $\mathcal{A}$, $\mathcal{J}$ and β_R. Hence, we are left with only 4 free real parameters: $\mathcal{A}$, $\mathcal{J}$, β_R, β_I.

Satisfying $\mathrm{Re}(\Upsilon_{5;2}) = 0$ and $\mathrm{Re}(\Upsilon_{6;3}) = 0$ is much more subtle. For this we turn to the integral representations (49) and (51) of the coefficients $\omega_{2;2}$ and $\omega_{3;3}$ of W. Consequently, one finds that

$$\mathrm{Re}(\Upsilon_{5;2}) = \frac{9}{32\pi}\sqrt{\frac{5}{\pi}}m^2\int_{\mathbb{R}^3}\langle Y_{20}|\partial_{rr}F(-x')\rangle\frac{1}{|x'|}d^3x' + \frac{37602}{199}m\mathcal{J}^2 = 0, \tag{54a}$$

$$\mathrm{Re}(\Upsilon_{6;3}) = \frac{3}{8\pi}\sqrt{\frac{7}{\pi}}\int_{\mathbb{R}^3}\langle Y_{30}|\partial_{rrr}F(-x')\rangle\frac{1}{|x'|}d^3x' + \frac{36399}{415}\sqrt{6}\mathcal{J}\beta_I + \frac{3808}{415}\sqrt{6}\mathcal{J}\delta_I = 0. \tag{54b}$$

Now, we note that $W = W[\mathcal{A},\mathcal{J},\beta_R,\beta_I](x)$. That is, the function W depends on the the choice of the free parameters $\mathcal{A}$, $\mathcal{J}$, β_R, β_I. For example, given our choice of boundary conditions, if $\mathcal{A} = \mathcal{J} = \beta_R = \beta_I = 0$, then the initial set reduces to that of the standard time symmetric, conformally flat slice of the Schwarzschild spacetime, and accordingly $W = m/2$. More crucially for our purposes, if $\mathcal{A} = \mathcal{A}_0 \neq 0$, but we keep $\mathcal{J} = \beta_R = \beta_I = 0$, then the initial data corresponds to a conformally flat, time asymmetric slice in the Schwarzschild spacetime, and —see [27]—

$$W = \frac{m}{2} - \frac{1}{576}\mathcal{A}^2|x|^2 + \mathcal{O}(|x|^4). \tag{55}$$

The latter fact suggests the use of an "implicit function theorem"-type of argument to see if there are some values of $\mathcal{A}$, $\mathcal{J}$, β_R and β_I such that (54a) and (54b) hold.

Given that the choice $\mathcal{A} = \mathcal{A}_0 \neq 0$, $\mathcal{J} = \beta_R = \beta_I = 0$ corresponds to Schwarzschild data, we have that

$$\mathrm{Re}(\Upsilon_{5;2})[\mathcal{A}_0,0,0,0] = 0, \tag{56a}$$

$$\mathrm{Re}(\Upsilon_{6;3})[\mathcal{A}_0,0,0,0] = 0; \tag{56b}$$

the smoothness of of static spacetimes at the critical sets where null infinity "touches" spatial infinity has been discussed in [18]. On the other hand, using lemma 1, $\mathrm{Re}(\Upsilon_{5;2})$ and $\mathrm{Re}(\Upsilon_{6;3})$ depend in, at least, C^1 fashion on $\mathcal{A}$. From (18) and (55) it follows that

$$\partial_{\mathcal{A}}\mathrm{Re}(\Upsilon_{5;2})[\mathcal{A}_0,0,0,0] \neq 0, \tag{57a}$$

$$\partial_{\mathcal{A}}\mathrm{Re}(\Upsilon_{6;3})[\mathcal{A}_0,0,0,0] \neq 0. \tag{57b}$$

Thus, if we set

$$\mathcal{F}(\mathcal{A},\mathcal{J},\beta_R,\beta_I) = \Big(\mathrm{Re}(\Upsilon_{5;2})[\mathcal{A},\mathcal{J},\beta_R,\beta_I]\Big)^2 + \Big(\mathrm{Re}(\Upsilon_{6;3})[\mathcal{A},\mathcal{J},\beta_R,\beta_I]\Big)^2, \tag{58}$$

one has that

$$\mathcal{F}(\mathcal{A}_0,0,0,0) = 0, \tag{59a}$$

$$\partial_{\mathcal{A}}\mathcal{F}(\mathcal{A}_0,0,0,0) \neq 0, \tag{59b}$$

so that from the implicit function theorem —see e.g. [1, 10]— there exists a neighbourhood $\mho \subset \mathbb{R}^3$, $(0,0,0) \in \mho$ and a C^1 function $\mathcal{A}(\mathcal{J},\beta_R,\beta_I)$, $\mathcal{A}(0,0,0) = \mathcal{A}_0$ such that

$$\mathcal{F}(\mathcal{A}(\mathcal{J},\beta_R,\beta_I),\mathcal{J},\beta_R,\beta_I) = 0, \quad (\mathcal{J},\beta_R,\beta_I) \in \mho. \tag{60}$$

The latter proves essentially proves our main result. A careful inspection reveals that $\Upsilon^-_{6;2} \neq 0$ if $J \neq 0$, $\beta_R \neq 0$ and $\beta_I \neq 0$. We note that a similar argument with the choice $\mathcal{A} = \mathcal{J} = \beta_R = \beta_I = 0$ does not work for it implies $\partial_{\mathcal{A}}\psi_{ab}\psi^{ab} = \partial_{\mathcal{A}}W = 0$ so that $\partial_{\mathcal{A}}\mathcal{F}(0,0,0,0) = 0$ —the implicit function theorem can not be applied. That is, the Schwarzschild initial data that one is perturbing have to be time necessarily asymmetric.

5 Conclusions

The initial data we have constructed in this article is a non-linear perturbation of time asymmetric Schwarzschild initial data. It is conformally flat and has non-vanishing angular momentum. Its time development should give rise to a radiative spacetime which could be interpreted as a distorted Schwarzschild black hole. The crucial remaining problem is to obtain an existence proof for the evolution equations which is valid up to (and beyond) the critical sets where null infinity "touches" spatial infinity —see the discussions in [17, 18]. Obtaining such a proof will require deeper insights into the structure of spatial infinity than the ones currently available.

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